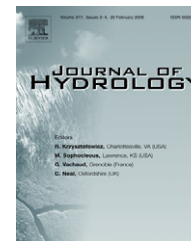




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A stochastic model of soil water regime in the crop root zone

Yi Luo ^{a,*}, Zhidong Lei ^b, Li Zheng ^{c,1}, Shixiu Yang ^b, Zhu Ouyang ^a,
Qianjun Zhao ^a

^a Institute of Geographical Sciences and Natural Resources Research, Chinese Academy of Sciences (CAS), Beijing, China

^b Department of Hydraulic Engineering, Tsinghua University, Beijing, China

^c Center for Agricultural Resources Research, Institute of Genetics and Developmental Biology, CAS, Beijing, China

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Summary The soil water regime in a crop root zone is critical to crop growth. Understanding the dynamics of the soil water regime is a prerequisite to proper irrigation management. However, due to the random nature of weather conditions, the soil water regime tends to be highly variable, which makes irrigation scheduling a difficult task. To better characterize the dynamic variability of soil water regime, we developed a stochastic model of soil water storage (SWS) by treating the evapotranspiration (ET) as an explicit random process. While developing this model, first of all, a root zone water balance model for SWS was established, parameterized, and validated with lysimeter data collected at Yucheng comprehensive experimental station (YCES) in Shandong Province, North China. We then employed 14 years of daily meteorological data collected at YCES to compute the daily reference evapotranspiration (ET_r) data series and performed time series analysis, established a discrete AR(1) model for ET_r and derived its continuous form by employing an even point sampling hypothesis. The stochastic model of SWS was formulated by incorporating the continuous AR(1) model into the deterministic model of SWS, which results in a system of two first-order temporal stochastic differential equations. Further, the Fokker–Planck equation of the probability density function (PDF) of SWS was derived and solved numerically. Consequently, the joint PDF of SWS and ET_r , the marginal PDF, mean, and deviation of SWS were obtained. These numerical solution results compare favorably with two years of SWS measurements. This indicates that the stochastic model can be a useful tool for irrigation scheduling and the associated risk assessment. © 2006 Published by Elsevier B.V.

* Corresponding author. Tel./fax: +86 10 6488 8920.

E-mail address: luoyi.cas@gmail.com (Y. Luo).

¹ Currently, a visiting scientist at the Mathematical modeling and numerical analysis group of the Los Alamos National Laboratory, USA.

Nomenclature**Notation**

A(1)	continuous first-order autoregressive model	ε_t	discrete white noise process of normal distribution
AR(1)	discrete first-order autoregressive model	$\mu(t)$	Weiner process
C	constant in soil water stress coefficient	σ_1	standard deviation of $\varepsilon(t)$
ET _r	reference evapotranspiration mm d ⁻¹	σ_2	standard deviation of $\varepsilon(t)$
I	irrigation density mm d ⁻¹	σ_{10}	initial standard deviation of S
K _c	crop coefficient	σ_{20}	initial standard deviation of V
K _{cm}	maximum value of crop coefficient	S ₀	initial mean of S
K _s	soil water stress coefficient	V ₀	initial mean of V
P	rainfall density mm d ⁻¹	Q	lower boundary flux of main root zone, mm d ⁻¹
S	dimensionless soil water storage	t	time, day of year
V	continuous form of the normalized residual of reference evapotranspiration	A(j), B(j), j = 1, 2	Fourier coefficients in ET _r mean analysis
V _t	discrete form of the normalized residual of reference evapotranspiration	SA(j), SB(j), j = 1, 2	Fourier coefficients in ET _r and STD analysis
W	soil water storage in root zone, mm	p	order of AR model
W _c	critical soil water storage defined in percolation formulation, mm	γ_k	partial correlation coefficient with lag time k days
W _f	field capacity, mm	ρ_k	Auto-regression coefficient with lag time k days
$\varepsilon(t)$	continuous white noise process of normal distribution	$\phi_j, j = 1, 2, \dots, p$	Auto-regression parameters in AR(p) model

12 Introduction

The soil water regime in the root zone is critical to crop growth. Understanding the dynamics of the soil water regime is a prerequisite to proper irrigation management. In practice, the root zone water balance equation is commonly used to predict soil water storage (SWS), which, at any time, changes with the net balance of inputs (precipitation, irrigation, upward movement of water to the root zone) and outputs (evapotranspiration and drainage) when the lateral flow can be neglected. Due to the uncertainty in knowledge of precipitation and evapotranspiration (ET), it is difficult to close the water balance by measurement, and subsequently the change of SWS is also a random process. Many efforts have been made to quantify the uncertainty of precipitation and ET and their role in modeling of SWS as a stochastic process. These efforts can generally be grouped into two types of approaches. The first approach takes into account only the randomness of precipitation in the root zone water balance equation while treating ET as a deterministic variable. Examples of this approach include Cordova and Bras (1979), Cordova and Bras (1981), Milly (1993), Rodriguez-Iturbe et al. (1991), Rodriguez-Iturbe et al. (1999), Laio et al. (2001), and Porporato et al. (2004). In situations where the uncertainty in estimating precipitation is insignificant and most of the randomness in SWS is caused by the fluctuation in other weather factors, such as wind speed, solar radiation, and temperature, many have adopted the second type of approach, in which precipitation is regarded as a deterministic variable while ET is modeled as stochastically. For example, Aboitiz et al. (1986) combined a stochastic representation of ET into the soil water balance equation and employed the state-space equation theory and Kalman filter to set up a comprehensive soil moisture estimation and forecasting framework. Similar work was also done by

Or and Groeneveld (1994) in stochastic estimation of plant-available soil water with a fluctuating ground water table. In the current paper, we study the stochastic characteristics of SWS in a heavily irrigated winter wheat field in North China Plain where an improved irrigation management program is needed. During the growing season of winter wheat in this region, from October till May next year, the precipitation is far less than crop evapotranspiration. We therefore adopt the second approach, consider precipitation a deterministic variable, and treat ET as the random process. In the works of Aboitiz et al. (1986) and Or and Groeneveld (1994), reference evapotranspiration, ET_r, was modeled with a discrete time series analysis and the SWS with a discrete state-space equation. Variance of the predicted SWS was obtained but the probability distribution was not. In irrigation scheduling, the characteristics of SWS, including its mean state, variation, and possible variation range at certain confidence levels, are all important to decision makers. Revealing the probability distribution of SWS and its evolution with time will help to understand the stochastic characteristics of the soil water regime. Therefore, the goal of the current paper is to (1) derive a stochastic model of SWS in the crop root zone; (2) compute the evolution of the probability distribution, mean, standard variation of SWS; and (3) reveal the stochastic behavior of SWS as influenced by the randomness of estimating evapotranspiration.

Model development**Conceptual model of SWS****Soil water balance equation**

The water balance equation is commonly used to estimate soil water storage in crop root zone. In arid or semi-arid

78 North China Plain, due to flat topography and insignificant
79 rainfall, the water balance equation can be simplified since
80 surface runoff and lateral soil water flow rarely happen in
81 cropped fields. Therefore, the soil water balance equation
82 in the root zone can be simply expressed as follows:

$$83 \frac{dW}{dt} = P + I - ET - Q, \quad (1)$$

86 where W is SWS (mm), P and I are the effective rainfall and
87 irrigation intensity (mm d^{-1}), ET is the actual evapotranspi-
88 ration rate of the crop (mm d^{-1}), Q is the soil water flux to
89 or from the crop root zone (mm d^{-1}), and t is time (d).

90 ET is often computed from the reference evapotranspira-
91 tion (ET_r), a dimensionless crop coefficient K_c and a soil
92 water stress coefficient K_s as (Allen et al., 1998)

$$93 ET = K_s K_c ET_r. \quad (2)$$

96 The crop coefficient is often expressed empirically as a
97 function of leaf area index (LAI) or crop development time.
98 The following formula is adopted here for K_c (Ouyang and
99 Luo, 2002)

$$100 K_c = K_{cm} \exp \left[-\frac{(t - t_m)^2}{C_m^2} \right], \quad (3)$$

103 where t_m is the time when K_c reaches its maximal value K_{cm} ,
104 and C_m is the shape factor. K_{cm} , t_m and C_m can be deter-
105 mined empirically from field experimental data.

106 K_s is often expressed as a function of the plant available
107 soil moisture content. Here, it is simply taken as function of
108 W (Luo et al., 1998)

$$110 K_s = \left(\frac{W}{W_f} \right)^c, \quad (4)$$

111 where W_f is soil water storage at field capacity in the root
112 zone (mm), and C is an empirical positive constant related
113 to soil properties.

114 Q represents the soil water flux at the lower boundary of
115 the root zone. A positive value indicates drainage and a neg-
116 ative value the upward movement of water into root zone.
117 The following empirical formula is adopted here for Q (Ouy-
118 ang and Luo, 2002):

$$119 Q = a \left(\frac{W}{W_f} \right)^b (W - W_c), \quad (5)$$

122 where a and b are empirical constants that can be derived
123 from experimental data, and W_c is a threshold SWS value re-
124 lated to soil properties that can also be determined with
125 field data.

126 Parameterization of soil water balance equation

127 In the current study, we employ measurements from a
128 weighting lysimeter to determine the parameters (K_c , K_s ,
129 and those in Q) of the water balance equation given in the
130 above section. This weighting lysimeter was constructed in
131 1990 at Yucheng Comprehensive Experimental Station
132 (YCES) in Shandong Province, North China. The lysimeter
133 was built in the middle of a $1.0 \times 10^6 \text{ m}^2$ cultivated field
134 and put into operation in 1991. The basic components of
135 the lysimeter are illustrated in Fig. 1. Component (I) is a
136 steel soil cylinder with a surface area of 3.14 m^2 and a soil
137 profile with depth of 4.5 m overlying 0.5 m of fine sand.

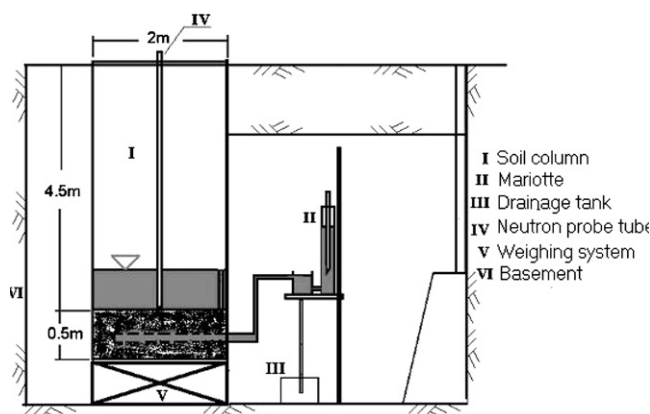


Figure 1 Diagram of the weighing lysimeter system.

The above ground part is 0.05 m in height. The steel cylinder
was inserted into the soil during the construction of the
lysimeter. Therefore, the lysimeter is filled with undisturbed
soil. A neutron probe access tube (IV) is installed in the col-
umn. The soil column rests on a sensitive weighing system (V)
which is capable of measuring the total mass up to 35 tons
with accuracy of $\pm 60 \text{ g}$. A Mariotte system (II) is connected
to the soil column to control and record the water table in-
side the lysimeter, and measure the amount of water that is
supplied to the soil column and/or leaks out of it. Gravity
drainage is collected by a drainage tank (III). The measure-
ments recorded in this lysimeter system include the weight
change of the soil column, water leakage from or water sup-
ply to the soil column, and the irrigation and/or rainfall
amount. The total ET , at certain time intervals, from this
lysimeter can be computed based on these measurements
through a water balance approach. Generally, observations
are made at 08:00 and 20:00 each day. The weighing system
is checked and recalibrated every year.

The soil moisture profile in the lysimeter was measured
with a neutron probe every five days and prior to and after
each irrigation or rainfall event. Lysimeter data collected
during the growth season of winter wheat of 1993, 1994,
and 1997 were checked for reliability and employed in this
work for model parameterization and validation. Daily
meteorological data from the past 14 years recorded in
YCES were used to calculate the reference evapotranspira-
tion with the Penman formula (Liu et al., 1997; Allen
et al., 1998).

Parameterization of Q was done using the lysimeter data
of 1993. The water balance zone was set as 1.0 m below the
surface ground, over which more than 90% of the winter
wheat roots were distributed (Luo et al., 2003). Field capac-
ity for the root zone was taken as 320.0 mm in this 1.0 m
layer (Ouyang and Luo, 2002). Flux at the lower boundary
of the main root zone was calculated with the observed
SWS and ET . Eq. (5) was then fitted with the calculated low-
er boundary flux to obtain the constants a and b by the least
square estimation method.

Using the actual ET measurements of the lysimeter, K_s
and K_c were jointly determined by least squares optimiza-
tion. Of the parameters defined in Eq. (3), K_{cm} was specified
as 1.15 based on our previous work (Ouyang and Luo, 2002);
 t_m was predetermined as 111 days according to LAI

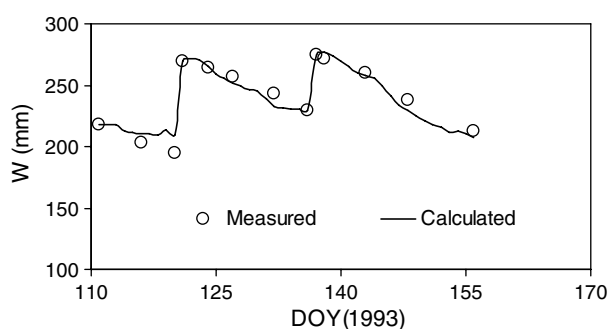


Figure 2 The measured and simulated SWS depletion processes (1993).

measurement results during the growing season of winter wheat. It was hypothesized here that K_c reaches its maximal value when LAI is the highest. C in K_s and C_m in K_c were also determined by the least square estimation method.

In summary, we obtained following parameters as $C = 0.50$, $a = 0.05$, $b = 2.77$, $W_f = 320.0$ mm, $W_c = 260.0$ mm, $t_m = 111$ d, $C_m = 45$, and $K_{cm} = 1.15$. Using these parameters, we calculated the SWS change from $t = 110$ d to $t = 155$ d in 1993. Fig. 2 gives the comparison between the observed and calculated SWS. The calculated SWS matched the observation points with an averaged relative error of 2% and the highest relative error of 7.1% on day 120.

The water balance model and the parameters determined in this section form the basis of further stochastic modeling of SWS. As we have explained before, in the current study, we model the randomness of SWS during the winter wheat growing season by primarily investigating the stochastic behavior of reference evapotranspiration. Details of stochastic modeling of reference ET, its incorporation into the water balance model, and the final model solution will be given in the following sections.

Time series analysis of the reference evapotranspiration (ET_r)

Efforts have been made to model stochastic evapotranspiration in previous works (Aboitiz et al., 1986; Or and Groeneveld, 1994), in which the daily reference evapotranspiration (ET_r) was described by a discrete moving average autoregressive model. To incorporate the stochastic model of ET_r into the continuous water balance model given in the previous sections, a continuous form of stochastic ET_r model is required. In the current section, we take the following steps to achieve this: (1) the seasonal trends of mean and standard deviation of ET_r were fitted with Fourier series; (2) the seasonal trends were removed and the residuals normalized with the mean; (3) discrete time series analyses were performed on the normalized residual series; and (4) with a hypothesis of fixed interval sampling of the daily ET_r , a continuous stochastic model of the residual ET_r was derived from the discrete residual series.

Mean and standard deviation of reference evapotranspiration ET_r

Daily reference evapotranspiration series of the past 14 years were employed here to investigate their stochastic

behavior. Fig. 3 plots estimates of ET_r , as calculated from weather records, showing a clear seasonal trend. ET_r goes up in spring time, reaches its maximum in summer, and then gradually decreases from late summer to winter. Fig. 4 shows the mean and standard deviation (STD) of ET_r .

Aboitiz et al. (1986) and Or and Groeneveld (1994) employed the first two-harmonics of a Fourier series to fit the mean and STD of ET_r as given by the following equation

$$\overline{ET_r(t)} = \overline{ET_r} + \sum_{j=1}^2 \left[A(j) \cos\left(\frac{2\pi jt}{365}\right) + B(j) \sin\left(\frac{2\pi jt}{365}\right) \right], \quad (6)$$

$$\overline{\sigma(t)} = \overline{\sigma} + \sum_{j=1}^2 \left[SA(j) \cos\left(\frac{2\pi jt}{365}\right) + SB(j) \sin\left(\frac{2\pi jt}{365}\right) \right], \quad (7)$$

where t is the Julian day, $\overline{ET_r(t)}$ is the mean of ET_r at day t , $\overline{ET_r}$ is the average of $ET_r(t)$ in a year, $\sigma(t)$ is the STD and $\overline{\sigma}$ its average in a year, $A(j)$, $B(j)$, $SA(j)$ and $SB(j)$ are the coefficients of the first two-harmonics of Fourier series, which can be determined by the least square estimation from the ET_r statistics as given in Fig. 4. For our case, fitting the parameters gives: $A(1) = -1.54$, $A(2) = -0.40$, $B(1) = 0.07$, $B(2) = -0.40$, $SA(1) = -0.41$, $SA(2) = -1.6$, $SB(1) = 0.32$, $SB(2) = -0.09$, $\overline{ET_r} = 3.72$ mm d⁻¹, $\overline{\sigma} = 0.48$ mm d⁻¹. There is a very good agreement between the fitted curves and the point mean and STD values of ET_r shown in Fig. 4.

Time series analysis of ET_r

To facilitate the time series analysis of $ET_r(t)$, here we define a new random variable V_t as given in equation (8) and perform time series analysis on V_t to formulate a continuous stochastic model of ET_r :

$$V_t = \frac{ET_r(t) - \overline{ET_r(t)}}{\sigma(t)}. \quad (8)$$

The mean and the standard variation of V_t are readily obtained as

$$E[V_t] = 0 \quad E[V_t V_t] = 1, \quad (9)$$

where $E[\]$ this is the expectation operator.

Discrete time series analysis of V_t

Here we adopt the classical time series analysis for fitting a stochastic model of the V_t series. The partial and auto-

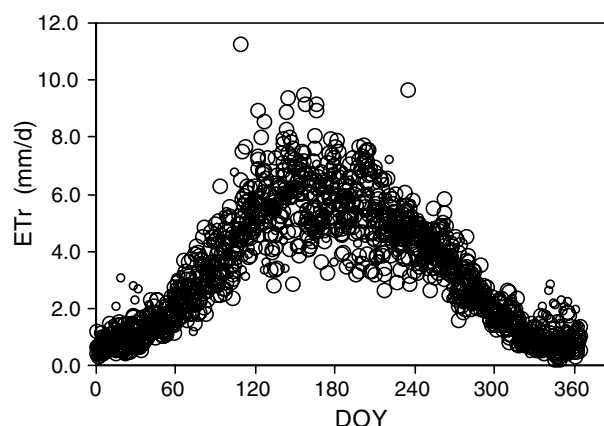


Figure 3 The calculated daily ET_r of 14 years.

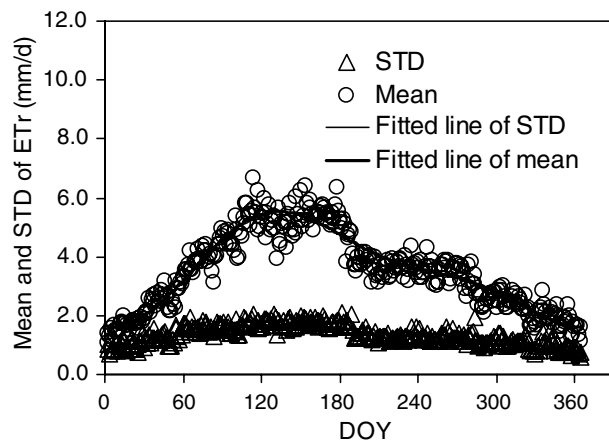


Figure 4 The mean, STD of the daily ET_r of 14 years and their fitted curves.

correlation coefficients of V_t were calculated with the following formula (An et al., 1983):

$$\gamma_k = \frac{1}{M} \sum_{l=0}^{M-k} V_l V_{l+k}, \quad (10)$$

$$\rho_k = \frac{\gamma_k}{\gamma_0}, \quad (11)$$

where γ_k and ρ_k are, respectively, the partial and auto-correlation coefficients calculated over a correlation time length of k day, and M is the number of sample points. The value of $\gamma_0 = 1.0$.

Fig. 5 gives a plot of the partial correlation function γ_k curve, which shows that it is truncated by an undetermined correlation time length p . Fig. 6 plots the auto-correlation function ρ_k , which decreases exponentially with correlation time. We therefore adopt an AR(p) model for the V_t series. The general form of an AR(p) model is given as (An et al., 1983)

$$V_t = \phi_1 V_{t-1} + \phi_2 V_{t-2} + \dots + \phi_p V_{t-p} + \varepsilon_t, \quad (12)$$

where ϕ_i , $i = 1, 2, \dots, p$, are the auto-regression parameters, p is the order of the AR models, and ε_t is a zero mean, normally distributed, random term. We adopt the moment estimate method to determine auto-regression parameters ϕ_i .

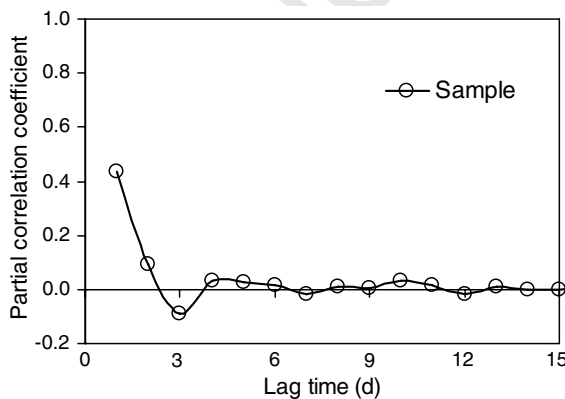


Figure 5 The partial correlation coefficient of the sample.

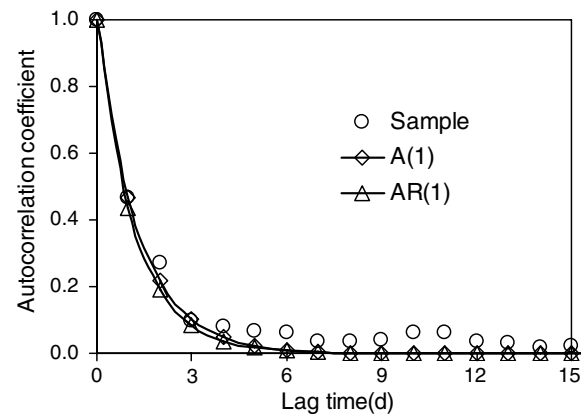


Figure 6 The autocorrelation coefficients of A(1) and AR(1) models and the sample.

The moment estimate method was chosen due to its simplicity and ease of application in actual crop water management. The values of ϕ_i , $i = 1, 2, \dots, p$ are determined by solving the following set of equations:

$$\begin{pmatrix} 1 & \rho_1 & \dots & \rho_{p-1} \\ \rho_1 & 1 & \dots & \rho_{p-2} \\ \dots & \dots & \dots & \dots \\ \rho_{p-1} & \rho_{p-2} & \dots & 1 \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \dots \\ \phi_p \end{pmatrix} = \begin{pmatrix} \rho_1 \\ \rho_2 \\ \dots \\ \rho_p \end{pmatrix}. \quad (13)$$

The order p can be determined through calculation of the AIC (Akaike Information Criterion; An et al., 1983) as given in Eq. (14). The smaller the AIC is, the better the AR model fits to the V_t series.:

$$AIC = \lg \sigma_1^2 + \frac{2p}{M}. \quad (14)$$

The deviation of ε_t , σ_1^2 is estimated by the following formula:

$$\sigma_1^2 = \gamma_0 - \sum_{j=1}^p \phi_j \gamma_j. \quad (15)$$

Table 1 lists the estimated parameters of AR(p), including the partial correlation coefficients γ_k , the estimated σ_1^2 , and the AIC values corresponding to different p values.

Procedures for selecting a particular model order have been given elsewhere, e.g., (Box and Jenkins, 1976; Salas et al., 1980). From Table 1, when $p = 1$ the AIC has the smallest value, hence an AR(1) model is chosen to represent the V_t series, and Eq. (12) can be rewritten as

$$V_t - \phi_1 V_{t-1} = \varepsilon_t. \quad (16)$$

The values of other parameters are $\phi_1 = 0.437$ and $\sigma_1^2 = 0.791$.

Continuous time series analysis of V_t

Given the daily V_t series as described by Eq. (16) and assuming that those daily series are samples of a fixed interval, the continuous form of AR(1), termed as A(1), can be described as (Pandit and Wu, 1983)

$$\frac{dV(t)}{dt} + \beta V(t) = \varepsilon(t), \quad (17)$$

Table 1 The estimated parameters of AR(p)

P	1	2	3	4
ϕ_1	0.437	0.094	-0.089	0.0321
σ_1^2	0.791	0.762	0.784	0.778
AIC	-0.087	-0.078	-0.061	-0.049

where $V(t)$ is the continuous form of the discrete V_t series, $\varepsilon(t)$ is a continuous, uncorrelated, normal distributed random term with a zero mean, and β is given by the following formula (Pandit and Wu, 1983:)

$$\beta = -\frac{\ln \phi_1}{\Delta}, \quad (18)$$

where Δ is the sampling interval of the random series. For the fixed interval daily V_t evapotranspiration series, Δ is simply taken as 1 day.

The random term $\varepsilon(t)$ can be expressed as the time differential of a Wiener process (Priestly, 1981; Wu et al., 1994)

$$\varepsilon(t) = \frac{d\mu(t)}{dt}, \quad (19)$$

where $d\mu(t)$ is the increment of Wiener process, and its variation is

$$E[d\mu(t)d\mu(t+dt)] = \sigma_2^2 dt = 2Ddt, \quad (20)$$

with D being the white noise density, and σ_2^2 being the variation of $\varepsilon(t)$ given as (Priestly, 1981)

$$\sigma_2^2 = \frac{2\beta\sigma_1^2}{1 - \phi_1^2}. \quad (21)$$

Therefore, the continuous A(1) model can be described by

$$dV(t) = -\beta V(t)dt + d\mu(t). \quad (22)$$

Eq. (22) is a typical linear Langevin equation that expresses a first order Markov process. Assuming the initial probability density distribution of $V(t)$ is

$$f(V, 0) = f_0(V). \quad (23)$$

The probability density function of $V(t)$ at time t has been derived by Wu et al. (1994) as

$$f(\varsigma, \chi) = \frac{1}{j\sqrt{2\pi[1 - \exp(-2\chi)]}} \int_{-\infty}^{\infty} \exp\left\{\frac{[\varsigma - \vartheta \exp(-\chi)]^2}{2[1 - \exp(-2\chi)]}\right\} f_0(\vartheta) d\vartheta, \quad (24)$$

where

$$\chi = \beta t, \quad \varsigma = jV\sqrt{\beta/D}. \quad (25)$$

When $f_0(V) = \delta(V - V_0)$, with V_0 being the initial value of $V(t)$, the probability density function of $V(t)$ becomes

$$f(V, t) = \frac{1}{\sqrt{2\pi\frac{D}{\beta}[1 - \exp(-2\beta t)]}} \times \exp\left\{-\frac{\beta[V - V_0 \exp(-\beta t)]^2}{2D[1 - \exp(-2\beta t)]}\right\}. \quad (26)$$

The mean and variance of V can then be readily obtained from Eq. (26) as

$$E[V] = V_0 \exp(-\beta t), \quad (27)$$

$$D[V] = \frac{D}{\beta}[1 - \exp(-2\beta t)]. \quad (28)$$

Both $E[V]$ and $D[V]$ are functions of time. As time t increases, $E[V]$ decreases and $D[V]$ increases exponentially. By defining two dimensionless variables $\tau_0 = 1/\beta$ and $t_s = \tau/\tau_0$, and using Eqs. (11), (15), (27) and (28), the following results can be obtained.

$$D = \gamma_0 \beta = \beta, \quad (29)$$

$$E[V] = V_0 \exp(-t_s), \quad (30)$$

$$D[V] = 1 - \exp(-2t_s). \quad (31)$$

As t equals to approximately $3\tau_0$, $E[V]$ will approach zero and $D[V]$ becomes unit 1, i.e., the probability density function of V approaches steady state: the standard normal distribution. This conclusion can be validated by plotting the accumulated probability distribution of the sampled V_t points (+) and the standard normal distribution (line), as shown in Fig. 7.

Stochastic modeling of SWS

The continuous form A(1) model of ET_r residual series derived in the previous section can now be easily incorporated into the mass balance equation of SWS established in the model development section. First, we replace V_t of the discrete Eq. (8) with $V(t)$ to obtain its continuous counterpart. Via rearrangement of its terms, we obtain

$$ET_r(t) = \overline{ET_r(t)} + \sigma(t)V. \quad (32)$$

The first and the second moment of V are

$$E[V] = 0 \quad E[VV] = 1. \quad (33)$$

Substituting Eq. (32) into (2), and then into (1) results in the following first-order stochastic differential equation with V as the random input

$$\frac{dW}{dt} = -[K_c(t)K_s(W)\overline{ET_r(t)} + Q - P - I] - K_c(t)K_s(W)\sigma(t)V \quad (34)$$

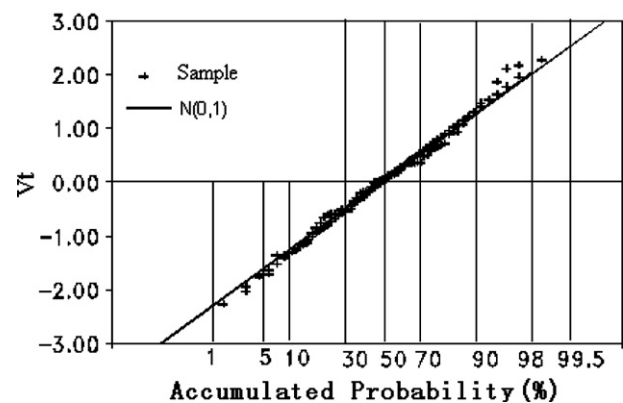


Figure 7 The accumulated probability of the V_t series and the normal distribution.

405 By defining a dimensionless variable S

$$407 \quad S = \frac{W}{W_f}, \quad (35)$$

408 and denoting

$$F(S, t) = -\frac{1}{W_f} [K_c(t)K_s(S)\overline{ET_p} + Q(S) - P - I], \quad (36)$$

$$410 \quad G(S, t) = -\frac{1}{W_f} K_c(t)K_s(S)\sigma. \quad (37)$$

411 We can rewrite Eq. (34) and combine it with Eq. (22) to
412 obtain the following differential equation set.

$$413 \quad dS = [F(S, t) + G(S, t)V]dt \quad (38)$$

$$415 \quad dV = -\beta V dt + d\mu(t)$$

416 Denoting the joint distribution density function of S and
417 V by $f(S, V, t)$, we can obtain the corresponding Fokker–
418 Planck equation of Eq. (38) in the sense of Ito as (Rodri-
419 guez-Iturbe et al., 1991; Wu et al., 1994).

$$420 \quad \frac{\partial f(S, V, t)}{\partial t} = -\frac{\partial}{\partial S} \{ [F(S, t) + G(S, t)V]f(S, V, t) \}$$

$$422 \quad + \frac{\partial}{\partial V} [\beta V f(S, V, t)] + D \frac{\partial^2}{\partial V^2} f(S, V, t). \quad (39)$$

423 We regard $S = 0$, $S = \infty$, and $V = \pm \infty$ as unrealistic condi-
424 tions, and hence set the boundary condition of Eq. (39) as

$$425 \quad f(0, \pm\infty, t) = f(\infty, \pm\infty, t) = 0. \quad (40)$$

428 Further, we hypothesize that S and V are distributed
429 independently at the initial time. When no uncertainty of
430 measurement error in SWS and ET_r is considered, the initial
431 condition of Eq. (39) can be given as

$$432 \quad f(S, V, 0) = \delta(S - S_0)\delta(V - V_0), \quad (41)$$

435 where δ is *delta Dirac* function, S_0 and V_0 are the mean of
436 S and V , respectively, at $t = 0$. When taking into account
437 the measurement uncertainties in SWS and ET_r and assum-
438 ing their initial distributions are approximately normal at
439 the range of interest and independent of each other,
440 the initial condition of Eq. (39) is more realistically given
441 as

$$442 \quad f(S, V, 0) = \frac{1}{\sqrt{2\pi\sigma_{10}\sigma_{20}}} \exp \left[-\frac{(S - S_0)^2}{2\sigma_{10}^2} - \frac{(V - V_0)^2}{2\sigma_{20}^2} \right], \quad (42)$$

445 where S_0 and V_0 are the mean of S and V respectively, and
446 σ_{10} and σ_{20} are the STD of S and V , respectively.

447 Eq. (39) with boundary condition equation (40) and initial
448 condition (41) or (42) can be solved numerically. From the
449 joint distribution of S and V and their marginal distribution
450 density function, the first and second moment of S can be
451 obtained through the following integration:

$$452 \quad f_s(S, t) = \int_{-\infty}^{+\infty} f(S, V, t) dV, \quad (43)$$

$$E[S, t] = \int_0^{\infty} S f_s(S, t) dS, \quad (44)$$

$$454 \quad D[S] = \int_0^{\infty} \{S - E[S, t]\}^2 f_s(S, t) dS, \quad (45)$$

455 where $f_s(S, t)$ is the marginal distribution function of S , $E[S]$
456 and $D[S]$ are the mean and variation of S , respectively, both
457 of which are time-dependent.

Results and discussion

458

459 The predictive algorithm as given in Eqs. (39)–(45) was ap-
460 plied to the simulation dynamics of soil water storage deple-
461 tion in the lysimeter of YCES during growing seasons of
462 winter wheat in year 1994 and year 1997, with rainfall and
463 irrigation as deterministic inputs. First we computed the
464 PDF of SWS numerically through an alternative direction im-
465 plicit finite-difference scheme (Lu and Guan, 1987; Lei
466 et al., 1988; Zill and Gullen, 2001), with parameters K_c ,
467 K_s , and Q as given in the model development section. The
468 numerical simulation started with using the measured SWS
469 as the mean of the initial estimate of S after being normal-
470 ized by field capacity. The measurement errors of SWS were
471 considered by setting the initial STD of S as 0.01, which cor-
472 responds to a standard deviation of 3.2 mm of SWS measure-
473 ments. Also the mean and the initial STD of V were taken as
474 zero and 1, respectively. In our calculation, we treated the
475 irrigation and the effective rainfall as deterministic and dis-
476 crete events, both of which would occur and finish at a spe-
477 cific time t . We obtained a PDF of SWS at the intervals
478 between two irrigation and/or rainfall events, and assumed
479 the PDF just prior to and after an irrigation/or rainfall event
480 follows the same distribution. We took into account the ef-
481 fects of irrigation or rainfall on S in the mean of S , i.e.

$$D[S, t]_{t=t^+} = D[S, t]_{t=t^-}, \quad (46)$$

$$f_s(S + IS, t^+) = f_s(S + IS, t^-), \quad (47)$$

$$E[S, t]_{t=t^+} = E[S, t]_{t=t^-} + IS, \quad (48) \quad 483$$

484 where t^- and t^+ symbolize left and right neighborhood of
485 time t , respectively, and IS is defined as

$$IS = P/W_f \text{ or } IS = I/W_f \quad (49) \quad 487$$

488 Subsequently, we derived the simulated mean and devi-
489 ation of W from that of S as

$$E[W, t] = W_f E[S, t], \quad (50)$$

$$D[W, t] = W_f^2 D[S, t]. \quad (51) \quad 491$$

492 Using the in situ determined parameter values, the fore-
493 casting of SWS using this model requires no other data ex-
494 cept the inputs of rainfall and irrigation events. To
495 evaluate the performance of the proposed stochastic mod-
496 el, we compared SWS measurements from the lysimeter in
497 YCES to the computed mean and STD of W for year 1994
498 and 1997. Fig. 8 (year 1994) and Fig. 9 (year 1997) gave
499 the predicted changes of the mean of W (heavy line), its
500 confidence limit at level $P = 95\%$ (light line), and the mea-
501 sured W (open circles). From these plots, we found that
502 fairly good agreement between simulation and measure-
503 ments with measured data points falling generally within
504 the confidence limits except towards the later part of year
505 1994. Therefore, we were confident that the initial condi-
506 tions and overall physical assumptions in the model con-
507 struction were reasonable. The execution of SWS
508 simulation required only the input of a W measurement at
509 the starting time, and involved no other W measurements.
510 This minimized the data requirements of this model and
511 provided convenience for its field application.

512 Figs. 8 and 9 also showed that the uncertainty of the SWS
513 prediction increased with lead-time of prediction. The
514 increase in uncertainty is typical for large lead-times (Or

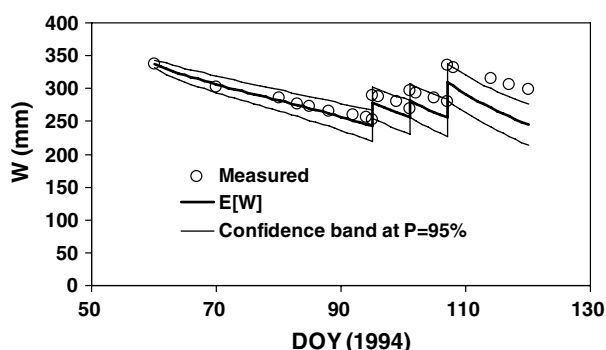


Figure 8 The measured points, predicted mean and confidence band at $P = 95\%$ of SWS for year 1994.

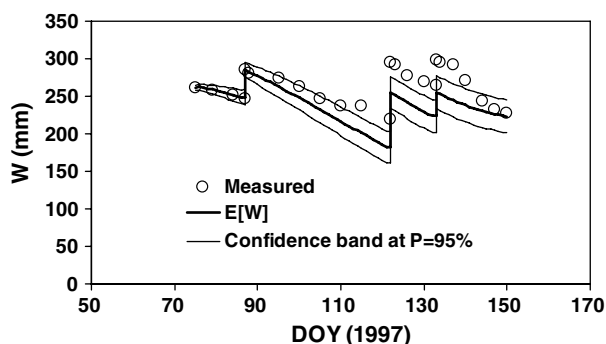


Figure 9 The measured points, predicted mean and confidence band at $P = 95\%$ of SWS for year 1997.

and Groeneveld, 1994; Aboitiz et al., 1986; and Luo et al., 1998). The predicted uncertainty may be used for risk analysis in making irrigation scheduling decisions. Depending on the needs of a particular application, the predictive algorithm we derived can give a confidence limit at any required confidence level. This is an obvious advantage of our model over that presented by Aboitiz et al. (1986) and Or and Groeneveld (1994). Depending on the availability of the measurement data, we performed predictions with look-ahead periods of 60 and 75 days in year 1994 and 1997, respectively. The predicted confidence band width of SWS at $P = 95\%$ at the end of the prediction periods are 19.8 mm in 1994 and 21.9 mm in 1997. Therefore the model performance is quite reliable even with such a long lead-time in prediction. For crop irrigation management, such a long lead-time is sufficient for any irrigation planning.

Concluding remarks

This paper presents a stochastic model for predicting the dynamic changes of soil water storage in the main root zone of crops for regions where the randomness in rainfall is insignificant. It provides a convenient tool for characterizing the soil water storage change and its variability for aiding decision making in scheduling irrigation. The stochastic model of ET_r adopted in current paper was limited to a continuous form of order 1 autoregressive model. If we treated the normalized residual part of ET_r as a continuous white noise process, the proposed stochastic model

will reduce to a simpler form as given in Luo et al. (1998). Further, it is also possible to adopt the methodology proposed by Graupe and Krause (1973) for transforming a more complex ARMA (1,1) process into an AR(1) model. Such a transformation may enable us to apply the proposed approach to describe a more complex stochastic process. When applying the proposed model to a specific site, stochastic modeling of reference evapotranspiration and parameterization of crop coefficients, soil water stress coefficient and percolation formulation need to be performed in advance. In addition, the uncertainty of soil water storage measurements should also be evaluated and tested. In the current work, the parameterization and validation of the proposed model were carried out on only a limited dataset at a specific site; further development should be done with different soils, crops, and weather conditions.

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